

Journal of Algebra and Its Applications
© World Scientific Publishing Company

ON THE TOPOLOGICAL RANK OF THE VARIETY OF RIGHT ALTERNATIVE METABELIAN LIE-NILPOTENT ALGEBRAS

ALEXEY KUZ'MIN*

*Institute of Math. and Stat., University of São Paulo
Rua do Matao, 1010 – Cidade Universitaria
São Paulo, São Paulo, 05508-090, Brazil
amkuzmin@ya.ru*

Received (Day Month Year)

Revised (Day Month Year)

Accepted (Day Month Year)

Communicated by (xxxxxxxxxx)

In 1981, S. V. Pchelintsev introduced the notion of topological rank for Spechtian varieties of algebras as a certain tool for studying the structure of non-nilpotent subvarieties in a given variety. We provide a variety of right alternative algebras of arbitrary given finite topological rank. Namely, we prove that the topological rank of the variety of right alternative metabelian (solvable of index two) algebras that are Lie-nilpotent of step not more than s over a field of characteristic distinct from two and three is equal to s .

Keywords: right alternative algebra, metabelian algebra, Lie-nilpotent algebra, superalgebra, variety of algebras, free algebra of variety, polynomial identity, Spechtian variety, Specht property of variety, topological rank of variety.

Mathematics Subject Classification 2010: 17D15, 17A50, 17A70.

1. Introduction

In 1986, A. R. Kemer [1,2] solved affirmatively the famous Specht problem [3] by proving that an arbitrary variety of associative algebras over a field of characteristic zero is finitely based. It is also known [4]–[6] that there are non-finitely based varieties of associative algebras over an arbitrary field of prime characteristic.

Recall that a variety of algebras is said to be *Spechtian* (or to have the *Specht property*) if its every subvariety is finitely based. The Kemer's theorem has certain analogs in the cases of Jordan, alternative, and Lie algebras over a field of characteristic 0. Namely, A. Ya. Vais, E. I. Zel'manov [7] proved the Specht property of the variety generated by Jordan PI-algebra on a finite set of generators. A. V. Iltyakov [8] obtained the similar result for alternative PI-algebras and also proved in [9] that

*The author is supported by the São Paulo Research Foundation (FAPESP) 2010/51880–2.

the variety generated by a finite dimensional Lie algebra is Spechtian. U. U. Umirbaev [10] proved the Specht property of every variety of solvable alternative algebras over a field of characteristic distinct from two and three. The questions about the Specht property for the varieties of all alternative, Lie, and Jordan algebras over a field of characteristic zero are still open problems.

Since 1976, it is known [11] that the variety of all right alternative metabelian algebras over an arbitrary field is not Spechtian. In 1985, I. M. Isaev [12] proved that non-finitely based varieties of right alternative metabelian algebras can even be generated by finite-dimensional algebras. The Specht property for certain varieties of right alternative algebras were also studied in [13]–[16].

Recall [17] the notion of topological rank for Spechtian varieties of algebras. Let \mathcal{V} be a Spechtian variety of algebras and \mathfrak{M} be a proper subvariety of \mathcal{V} . By a *system distinguishing \mathfrak{M} from \mathcal{V}* we mean a set $S = \{f_1, \dots, f_n\}$ of nonzero homogeneous polynomials of the free \mathcal{V} -algebra such that \mathfrak{M} can be defined by the union of the identities $f_1 = 0, \dots, f_n = 0$ with the defining identities of \mathcal{V} . A *degree of the system S* is the maximal degree of its polynomials. A *dimension $\dim_{\mathcal{V}} \mathfrak{M}$ of \mathfrak{M} with respect to \mathcal{V}* is the minimal possible degree of a system distinguishing \mathfrak{M} from \mathcal{V} . Let $\wp(\mathcal{V})$ be the *set of all subvarieties of \mathcal{V}* . For every $\mathfrak{R} \in \wp(\mathcal{V})$ and $n \in \mathbb{N}$ by $\mathring{U}_n(\mathfrak{R})$ we denote the *set of all proper subvarieties $\mathfrak{M} \subset \mathfrak{R}$ of dimensions $\dim_{\mathfrak{R}} \mathfrak{M} \geq n$* and put $U_n(\mathfrak{R}) = \mathring{U}_n(\mathfrak{R}) \cup \{\mathfrak{R}\}$. By definition, it is clear that

$$U_n(\mathfrak{R}) \cap U_{n'}(\mathfrak{R}) = U_{\max(n, n')}(\mathfrak{R})$$

and, for every $\mathfrak{M} \in \mathring{U}_n(\mathfrak{R})$, we have $U_n(\mathfrak{M}) \subset U_n(\mathfrak{R})$. Therefore considering a set

$$\mathfrak{B} = \{U_n(\mathfrak{R}) \mid \mathfrak{R} \in \wp(\mathcal{V}), n \in \mathbb{N}\}$$

as a base for the neighborhoods, we endow $\wp(\mathcal{V})$ with a topology. Every subset $\wp(\mathcal{V})$ gains a structure of the topological space with respect to the induced topology. For every $\Omega \subseteq \wp(\mathcal{V})$ by Ω' denote a *subspace of Ω obtained by the exclusion of all its isolated points*. Note that by virtue of the Specht property of \mathcal{V} , every descending chain of varieties in $\wp(\mathcal{V})$ stabilizes and, consequently, every topological subspace of $\wp(\mathcal{V})$ contains isolated points. Therefore one can consider the strictly descending chain

$$\Omega \supset \Omega' \supset \Omega'' \supset \dots \supset \Omega^{(n)} \supset \dots$$

The *topological rank $r_t(\Omega)$ of the space Ω* is the minimal n such that $\Omega^{(n)} = \emptyset$ or \mathfrak{R}_0 if such an n doesn't exist. The value $r_t(\wp(\mathcal{V}))$ is called the *topological rank of the variety \mathcal{V}* and is denoted shortly by $r_t(\mathcal{V})$. For example, if \mathcal{V} is nilpotent, then every subvariety of \mathcal{V} turns out to be an isolated point of $\Omega = \wp(\mathcal{V})$ and, consequently, $r_t(\mathcal{V}) = 1$. Otherwise, Ω' consists of all non-nilpotent subvarieties of \mathcal{V} and its isolated points are the varieties that were limit points in Ω for the sequences of nilpotent varieties only. Further, if Ω'' is not empty, then its isolated points are the varieties that were limit points in Ω' for the sequences containing only isolated points of Ω' , etc.

The structures of a set of non-nilpotent subvarieties for various varieties of nearly associative metabelian algebras were studied in [17]–[19]. A. V. Badeev [20] provided a chain $\mathcal{V}_1 \subset \dots \subset \mathcal{V}_n \subset \dots \subset \mathcal{V}$ of varieties of commutative alternative nil-algebras over a field of characteristic three such that $r_t(\mathcal{V}_n)$ is a linear function on n and $r_t(\mathcal{V}) = \aleph_0$. In 2007, S. V. Pchelintsev [16] constructed a variety \mathfrak{M} of right alternative metabelian algebras of almost finite topological rank, i. e. a variety \mathfrak{M} such that $r_t(\mathfrak{M}) = \aleph_0$ and $r_t(\mathfrak{M}')$ is finite for every proper subvariety $\mathfrak{M}' \subset \mathfrak{M}$.

Formulation of the result

Let F be a field of characteristic $\text{char}(F) \neq 2, 3$ and RA_2 be the variety of right alternative metabelian algebras over F defined by the identities

$$(x, y, z) + (x, z, y) = 0 \quad (\text{the right alternative identity}), \quad (1.1)$$

$$(xy)(zt) = 0 \quad (\text{the metabelian identity}), \quad (1.2)$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of the variables x, y, z . By $\text{RA}_2^{(s)}$ we denote the subvariety of RA_2 distinguished by the identity

$$\left[\dots [x_1, x_2], \dots, x_s, x_{s+1} \right] = 0 \quad (1.3)$$

of Lie-nilpotency of step s , where $[x, y] = xy - yx$ is the commutator of x, y .

The Specht property of $\text{RA}_2^{(s)}$ is proved by the author in [15]. By virtue of nilpotency of every commutative subvariety of RA_2 , we have $r_t(\text{RA}_2^{(1)}) = 1$. S. V. Pchelintsev established in [17] that $r_t(\text{RA}_2^{(2)}) = 2$. In the present paper, we prove the following

Theorem. *The topological rank of the variety $\text{RA}_2^{(s)}$ is equal to s for all natural s .*

The paper is organized as follows. In Sec. 2, we provide some preliminary results about the free RA_2 -algebra $F_{\text{RA}_2}[X]$ on a countable set X of generators over F . Sec. 3 is devoted to the studying of relations of the free algebra $F_{\text{RA}_2^{(s)}}[X]$. In Sec. 4, we construct a system of linear generators for the space of multilinear polynomials in $F_{\text{RA}_2^{(s)}}[X]$ of sufficiently high degree and obtain the upper bound $r_t(\text{RA}_2^{(s)}) \leq s$ by estimating the values of topological ranks of some subvarieties in $\text{RA}_2^{(s)}$ of special type. Finally, in Sec. 5, we construct an auxiliary $\text{RA}_2^{(s)}$ -superalgebras and considering the identities of their Grassmann envelopes obtain the low bound $r_t(\text{RA}_2^{(s)}) \geq s$.

2. Preliminaries

Throughout the paper, F is a field of characteristic $\text{char}(F) \neq 2, 3$; all vector spaces (algebras, superalgebras) are considered over F ; $X = \{x_1, x_2, \dots\}$ is a countable set; $\mathfrak{A} = F_{\text{RA}_2}[X]$ is a free RA_2 -algebra on the set X of generators; R_x and L_x are, respectively, the operators of right and left multiplication by the element x ; $H_x = R_x - L_x$; \mathfrak{A}^* is the associative algebra generated by all the operators R_x and

4 Alexey Kuz'min

L_x , for $x \in \mathfrak{A}$, acting on \mathfrak{A}^2 and by the identical mapping id ; $\text{Var } A$ is the variety generated by an algebra A .

Recall [15,16] that \mathfrak{A}^* satisfies the relations

$$R_x^2 = 0, \quad (2.1)$$

$$[R_x R_y, L_z] = 0, \quad (2.2)$$

$$[R_x, L_y] = -L_x L_y. \quad (2.3)$$

Relations (2.1), (2.2) imply immediately the following

Lemma 2.1. *The operator $R_x R_y$ lies in the center of \mathfrak{A}^* .*

Proposition 2.1. *The algebra \mathfrak{A}^* satisfies the relation*

$$3R_x R_y + H_x H_y = 2[R_x, H_y] + H_x R_y + H_y R_x. \quad (2.4)$$

Proof. Using (2.3), we have

$$\begin{aligned} H_x H_y &= (R_x - L_x)(R_y - L_y) = R_x R_y + L_x L_y - L_x R_y - R_x L_y = \\ &= R_x R_y - [R_x, L_y] - L_x R_y - L_y R_x - [R_x, L_y] = \\ &= R_x R_y - 2[R_x, L_y] - L_x R_y - L_y R_x. \end{aligned}$$

Combining the obtained relation with (2.1) and (2.3), we get

$$\begin{aligned} 3R_x R_y + H_x H_y &= 4R_x R_y - 2[R_x, L_y] - L_x R_y - L_y R_x = \\ &= 2[R_x, R_y] - 2[R_x, L_y] + (H_x - R_x) R_y + (H_y - R_y) R_x = \\ &= 2[R_x, H_y] + H_x R_y + H_y R_x. \quad \square \end{aligned}$$

In what follows, we use the symbol T as a common notation for the operator symbols R and H . The notation $w = T_x \dots T_y$ means that each operator symbol of the word w can be equal to R or H independently. In the case when all operator symbols in some word are assumed to be equal to each other, we use the notation

$$T(i_1, \dots, i_n) = \begin{cases} R_{x_{i_1}} \dots R_{x_{i_n}}, & \text{if } T = R, \\ H_{x_{i_1}} \dots H_{x_{i_n}}, & \text{if } T = H \end{cases}$$

and set $T(\emptyset) = \text{id}$.

Lemma 2.2. *The algebra \mathfrak{A}^* is spanned by the operators*

$$H(i_1, \dots, i_n) R(j_1, \dots, j_m).$$

Proof. Let \mathcal{I} be a linear span of all operators $H(i_1, \dots, i_n) R(j_1, \dots, j_m)$. It suffices to prove the inclusions $R(k)\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}H(k) \subseteq \mathcal{I}$. Note that (2.4) yields $R(i)H(j) \in \mathcal{I}$. Hence the inclusion $R(k)\mathcal{I} \subseteq \mathcal{I}$ can be easily proved by induction on the length of the operator $H(i_1, \dots, i_n)$. At the same time, Lemma 2.1 implies $\mathcal{I}H(k) \subseteq \mathcal{I}$. \square

Let \mathcal{L} be a linear span in \mathfrak{A}^* of all operators of the form

$$L_{x_i} w, \quad w \in \mathfrak{A}^*.$$

By virtue of (2.3), \mathcal{L} forms an ideal of \mathfrak{A}^* and by induction on n one can prove the congruence

$$H(1, \dots, n) \equiv R(1, \dots, n) \pmod{\mathcal{L}}, \quad n \in \mathbb{N}. \quad (2.5)$$

3. Relations of the free $\mathbf{RA}_2^{(s)}$ -algebra

Let $\mathfrak{A}_s = F_{\mathbf{RA}_2^{(s)}}[X]$ be the free $\mathbf{RA}_2^{(s)}$ -algebra on the set X of generators. Lemma 2.2 implies immediately the following

Lemma 3.1. *The linear span of all operators of degree $d \geq s$ in \mathfrak{A}_s^* is spanned by the operators*

$$H(i_1, \dots, i_n) R(j_1, \dots, j_{d-n}), \quad n < s.$$

In what follows, the term "polynomial" means a homogeneous polynomial of degree not less than two.

Definition 3.1. Let \approx be a symmetric relation on the set of polynomials of \mathfrak{A} such that $f_0 \approx f_1$ if $f_i = f_{1-i} R(j_1, \dots, j_{2k})$, $i \in \{0, 1\}$, and f_{1-i} doesn't depend on the variables $x_{j_1}, \dots, x_{j_{2k}}$. By the same symbol \approx we denote the induced relation on \mathfrak{A}^* : $\xi \approx \eta$ for $\xi, \eta \in \mathfrak{A}^*$ if $(x_i x_j) \xi \approx (x_i x_j) \eta$ and ξ, η do not depend on x_i, x_j .

Proposition 3.1. *The algebra \mathfrak{A}_s satisfies the relation*

$$x^3 \approx 0. \quad (3.1)$$

Proof. Using (1.1) and (1.2), we have

$$2yx^3 = y(x \circ x^2) = (yx^2)x = ((yx)x)x = (yx)x^2 = 0.$$

Hence, $x^3 \mathcal{L} = 0$. Therefore applying (2.5), for even $n \geq s$, and taking into account (1.3), we obtain

$$x^3 \approx x^3 R(1, \dots, n) = x^3 H(1, \dots, n) = 0. \quad \square$$

We say that *almost all polynomials of \mathfrak{A}_s (operators of \mathfrak{A}_s^*) satisfy some condition ϑ* if there is a natural n such that ϑ holds for all polynomials (operators) of degree more than n .

Lemma 3.2. *If $f \approx 0$ for $f \in \mathfrak{A}_s$, then almost all operators of \mathfrak{A}_s^* annihilate f .*

Proof. Assume that $f R(j_1, \dots, j_{2k}) = 0$, where f doesn't depend on $x_{j_1}, \dots, x_{j_{2k}}$. In view of Lemma 3.1, every operator word $\xi \in \mathfrak{A}_s^*$ of the degree $d \geq s + 2k$ can be represented as

$$\xi = \eta R(j_1, \dots, j_{2k}), \quad \eta \in \mathfrak{A}_s^*.$$

6 Alexey Kuz'min

Hence by Lemma 2.1, we have

$$f\xi = f R(j_1, \dots, j_{2k})\eta = 0. \quad \square$$

Lemma 3.3. *Almost all operators of \mathfrak{A}_s^* are skew-symmetric with respect to all their variables.*

Proof. We set $w \in \mathfrak{A}_s^2$. By virtue of (1.2) and (2.1), the partial linearization (see [21, Chap. 1]) of (3.1) has the form

$$(wx)x + (xw)x + x^2w = (xw)x \approx 0,$$

whence, $H_x R_x = -L_x R_x \approx 0$. Thus in view of Lemma 3.1, it remains to calculate

$$H_x H_x \approx H_x H_x R_y R_z \approx -H_x H_y R_x R_z = -H_x R_x R_z H_y \approx 0. \quad \square$$

Proposition 3.2. *The algebra \mathfrak{A}_s satisfies the relation*

$$(xy)T_x T_y \approx 0. \quad (3.2)$$

Proof. By virtue of Lemmas 2.1, 3.3, it suffices to verify that $(xy)R_x R_y \approx 0$. Using (2.1), (1.1), and Lemma 3.3, we have

$$(xy)R_x R_y = -(xy)R_y R_x = y^2 L_x R_x \approx 0. \quad \square$$

Proposition 3.3. *The algebra \mathfrak{A}_s^* satisfies the relations*

$$3R_x R_y - 2[R_x, H_y] + H_x H_y \approx 0, \quad (3.3)$$

$$[R_x, H_y H_z] \approx 0. \quad (3.4)$$

Proof. By Lemma 3.3, relation (3.3) follows from (2.4). Using (3.3) and combining Lemmas 2.1, 3.3 with the Jacobian identity, we obtain

$$\begin{aligned} [R_x, H_y H_z] &\approx 2[R_x, [R_y, H_z]] \approx \\ &\approx [R_x, [R_y, H_z]] - [R_y, [R_x, H_z]] = [H_z, [R_y, R_x]] = 0. \quad \square \end{aligned}$$

Definition 3.2. Let \mathcal{I} be an ideal of \mathfrak{A}_s^* . For $\xi, \eta \in \mathfrak{A}_s^*$ we write $\xi \cong \eta \pmod{\mathcal{I}}$ if there is a $\theta \in \mathcal{I}$ such that $\xi - \eta \approx \theta$.

Let \mathcal{H}_n ($n < s$) be the ideal of \mathfrak{A}_s^* generated by all the elements $H(i_1, \dots, i_n)$.

Proposition 3.4. *The algebra \mathfrak{A}_s^* satisfies the relation*

$$H(1, \dots, 2t) \cong 0 \pmod{\mathcal{H}_{2t+1}}. \quad (3.5)$$

Proof. We set $\eta = H(1, \dots, 2t)$. Applying (3.3) and (3.4), we have

$$3\eta \approx 3\eta R_x R_y \cong 2\eta R_x H_y \approx 2R_x \eta H_y \cong 0 \pmod{\mathcal{H}_{2t+1}}. \quad \square$$

4. Upper bound for the topological rank of $\mathbf{RA}_2^{\langle s \rangle}$

Definition 4.1. An n -allotted variety ($1 \leq n \leq s$) is a subvariety \mathcal{V} of $\mathbf{RA}_2^{\langle s \rangle}$ such that the free \mathcal{V} -algebra on the set X of generators satisfies the relation

$$\varphi(x_1, \dots, x_{n+1}) \approx 0, \quad (4.1)$$

where

$$\varphi(x_1, \dots, x_{n+1}) = \begin{cases} [\dots [x_1, x_2], \dots, x_n], x_{n+1}], & \text{if } n \text{ is even,} \\ [\dots [x_1 x_2, x_3], \dots, x_n], x_{n+1}], & \text{if } n \text{ is odd.} \end{cases}$$

By definition, every 1-allotted variety \mathfrak{M} is right nilpotent. Moreover, applying Lemma 3.1, it is not hard to prove that \mathfrak{M} is nilpotent and, consequently, $r_t(\mathfrak{M}) = 1$. We also stress that the variety $\mathbf{RA}_2^{\langle s \rangle}$ is s -allotted: for even s , it is clear by definition and, for odd s , it follows from (3.5).

Let \mathcal{V} be an n -allotted variety ($n \geq 2$), \mathcal{A} be the free \mathcal{V} -algebra on the set X of generators, and $\mathcal{P}_{d,n}$ ($d \geq 3$) be the subspace of multilinear polynomials in \mathcal{A} on the variables x_1, \dots, x_d . In order to avoid complicated formulas while writing down the polynomials of $\mathcal{P}_{d,n}$ we omit the indices of variables at the operator symbols and assume them to be arranged at the ascending order. For example, the notation $w = (x_2 x_5) H^2 R^3$ means the monomial

$$w = (x_2 x_5) H(1, 3) R(4, 6, 7).$$

Definition 4.2. Regular words are the polynomials of $\mathcal{P}_{d,n}$ of the following types:

- 1) $(x_1 \circ x_i) H^{2j} R^{d-2j-2},$
- 2) $[x_1, x_i] H^{2j} R^{d-2j-2},$
- 3) $[x_2, x_3] H^{2j} R^{d-2j-2},$
- 4) $[x_1, x_2] H^{2k-1} R^{d-2k-1},$

where $i = 2, 3, \dots, d$; $j = 0, 1, \dots, t-1$; $k = 1, 2, \dots, n-t-1$; $t = \lfloor \frac{n}{2} \rfloor$.

Lemma 4.1. Almost all polynomials of $\bigcup_{d=3}^{\infty} \mathcal{P}_{d,n}$ are linear combinations of regular words.

Proof. By Lemma 3.3, there is a degree d such that every monomial

$$(x_1 x_2) T_3 \dots T_d \in \mathcal{P}_{d,n}$$

is skew-symmetric w.r.t. x_3, \dots, x_d . Consequently, in view of Lemma 3.1 and relation (3.5), $\mathcal{P}_{d,n}$ can be spanned by the polynomials

$$(x_i \circ x_j) H^k R^{d-k-2}, \quad [x_i, x_j] H^k R^{d-k-2},$$

where $a \circ b = ab + ba$, $1 \leq i < j \leq d$, and $k = 0, 1, \dots, 2t-1$.

8 *Alexey Kuz'min*

Linearizing (3.1) and (3.2), we have

$$\begin{aligned} (x \circ y) T_z + (y \circ z) T_x + (z \circ x) T_y &\approx 0, \\ [x, y] T_z T_t + [x, t] T_z T_y + [z, y] T_x T_t + [z, t] T_x T_y &\approx 0. \end{aligned}$$

Applying these relations, it is not hard to prove that $\mathcal{P}_{d,n}$ can be spanned by the polynomials:

$$\begin{aligned} 1') & (x_1 \circ x_i) H^k R^{d-k-2}, \\ 2') & [x_1, x_i] H^k R^{d-k-2}, \\ 3') & [x_2, x_3] H^k R^{d-k-2}, \end{aligned}$$

where $i = 2, \dots, d$ and $k = 0, 1, \dots, 2t - 1$.

By \mathcal{W} denote the linear span of all regular words of types 1)–3). Note that the polynomials of types 1')–3') lie in \mathcal{W} for even k . Let us verify that the polynomials of types 1')–3') for odd k can be represented as linear combinations of regular words.

By virtue of (1.1), we have

$$(x \circ y) H_z = (x \circ y) z - z(x \circ y) = (x \circ y) z - (zx)y - (zy)x.$$

Hence, in view of (3.4), every polynomial of type 1') lie in \mathcal{W} . Further, using Lemmas 2.1, 3.3, the partial linearization

$$(xy)y + (yx)y + y^2x \approx 0$$

of (3.1), identity (1.1) and relation (3.3), we get

$$\begin{aligned} [x, y] H_y &\approx [x, y] H_y R_z R_u \approx [x, y] R_z R_u H_y = [x, y] R_y R_z H_u = \\ &= ((xy)y - (yx)y) R_z H_u \approx (2(xy)y + y^2x) R_z H_u = y^2(2L_x + R_x) R_z H_u = \\ &= y^2(3R_x - 2H_x) R_z H_u \approx y^2(2[R_x, H_z] - H_x H_z - 2H_x R_z) H_u \approx \\ &\approx y^2(2R_x - H_x) H_z H_u = y^2(R_x + L_x) H_z H_u = (y^2 R_x + (xy) R_y) H_z H_u. \end{aligned}$$

In view of (3.4) the obtained relation implies that the polynomials of types 2'), 3') for odd k are skew-symmetric modulo \mathcal{W} with respect to all their variables. Therefore every polynomial of type 2'), 3') is proportional modulo \mathcal{W} to a regular word of type 4). \square

Lemma 4.2. *For every n -allotted variety \mathcal{V} ($n \geq 2$) there is a punctured neighborhood $\mathring{\mathcal{U}}_d(\mathcal{V})$ such that every variety of $\mathring{\mathcal{U}}_d(\mathcal{V})$ is $(n-1)$ -allotted.*

Proof. By virtue of the restriction $\text{char}(F) \neq 2$, Lemma 3.3 and relation (3.1) imply that in some punctured neighborhood of \mathcal{V} every variety can be defined by a system of identities where all polynomials starting from some sufficiently high degree are multilinear. Consequently by Lemma 4.1, we can choose a punctured neighborhood $\mathring{\mathcal{U}}_d(\mathcal{V})$ such that every variety $\mathfrak{M} \in \mathring{\mathcal{U}}_d(\mathcal{V})$ satisfies an identity $f = 0$, where f is a nontrivial linear combination of regular words of $\mathcal{P}_{d,n}$. Let \mathcal{A} be the

free \mathfrak{M} -algebra on the set X of generators. We write down relation (4.1) shortly as $\mathcal{A}^2 H^{2t} \approx 0$ if $n = 2t + 1$ and as $\mathcal{A} H^{2t} \approx 0$ if $n = 2t$.

First consider the case $n = 2t + 1$. We prove that relation $\mathcal{A}^2 H^{2t} \approx 0$ and identity $f = 0$ imply $\mathcal{A} H^{2t} \approx 0$. By Lemma 4.1, f can be presented in the form

$$f \equiv \sum_{i=2}^d \sum_{j=0}^{t-1} \left(\alpha_{2j}^{(i)} (x_1 \circ x_i) H^{2j} R^{d-2j-2} + \alpha_{2j+1}^{(i)} [x_1, x_i] H^{2j} R^{d-2j-2} \right) \pmod{\mathcal{W}_{3,4}},$$

where $\mathcal{W}_{3,4}$ is the linear span of regular words of types 3), 4). We fix $i \geq 4$ and a minimal index ℓ such that $\alpha_\ell^{(i)} \neq 0$. Then by the substitution $x_i := aH^{2t-\ell}$, for $a \in \mathcal{A}$, using the equality $R_y + L_y = 2R_y - H_y$ and relation (3.4), we obtain $aH^{2t} \approx 0$. Otherwise, we can rewrite f in the form

$$f = \sum_{k=0}^{t-1} g_k + \sum_{k=1}^t h_k,$$

where

$$g_0 = \left(\alpha_0 [x_1, x_2] x_3 + \beta_0 [x_3, x_1] x_2 + \gamma_0 [x_2, x_3] x_1 \right) R^{d-3},$$

$$h_t = \zeta_t [x_1, x_2] H^{2t-1} R^{d-2k-1},$$

and

$$g_k = \left(\alpha_k [[x_1, x_2], x_3] + \beta_k [[x_3, x_1], x_2] + \gamma_k [[x_2, x_3], x_1] \right) H^{2k-1} R^{d-2k-2},$$

$$h_k = \delta_k (x_1 \circ x_2) H^{2k} R^{d-2k-2} + \varepsilon_k (x_1 \circ x_3) H^{2k} R^{d-2k-2} + \zeta_k [x_1, x_2] H^{2k-1} R^{d-2k-1},$$

for $k = 1, \dots, t-1$. If at least one of the coefficients $\alpha_0, \beta_0, \gamma_0$ is not zero, then by three successive substitutions $x_i := aH^{2t-1}$ ($i = 1, 2, 3$), we have

$$\begin{cases} (\alpha_0 + \beta_0) aH^{2t} \approx 0, \\ (\alpha_0 + \gamma_0) aH^{2t} \approx 0, \\ (\beta_0 + \gamma_0) aH^{2t} \approx 0. \end{cases}$$

Hence in view of the restriction $\text{char}(F) \neq 2$, we obtain either $aH^{2t} \approx 0$ or $g_0 = 0$. Further, if $\varepsilon_1 \neq 0$, then by the substitution $x_3 := aH^{2t-2}$, we have $aH^{2t} \approx 0$. Otherwise, if at least one of the coefficients δ_1 or ζ_1 is not zero, by two successive substitutions $x_i := aH^{2t-2}$ ($i = 1, 2$), we obtain

$$\begin{cases} (2\delta_1 + \zeta_1) aH^{2t} \approx 0, \\ (2\delta_1 - \zeta_1) aH^{2t} \approx 0. \end{cases}$$

Thus we have either $aH^{2t} \approx 0$ or $h_1 = 0$ and, consequently, f gets the form

$$f = \sum_{k=1}^{t-1} g_k + \sum_{k=2}^t h_k.$$

10 *Alexey Kuz'min*

Therefore by the same arguments as above, we obtain either $aH^{2t} \approx 0$ or

$$g_1 = h_2 = \cdots = g_{t-2} = h_{t-1} = g_{t-1} = 0$$

and

$$f = h_t = \zeta_t [x_1, x_2] H^{2t-1} R^{d-2t-1}.$$

But in this case, the assumption of the lemma implies $\zeta_t \neq 0$ and, consequently, $\mathcal{A}H^{2t} \approx 0$.

Now consider the case $n = 2t$. We need to prove that relation $\mathcal{A}H^{2t} \approx 0$ and identity $f = 0$ imply $\mathcal{A}^2 H^{2t-2} \approx 0$. Unlike the case $n = 2t + 1$, the regular word of type 4) corresponding to the index $k = t$ vanishes. All the other regular words are the same. Therefore by the similar arguments as above, reducing per unit the power $p(t)$ for every substitution $x_i := aH^{p(t)}$ and assuming $a \in \mathcal{A}^2$, one can prove that $\mathcal{A}^2 H^{2t-1} \approx 0$. By (3.5), the obtained relation yields $\mathcal{A}^2 H^{2t-2} \approx 0$. \square

As it was stressed above, every 1-allotted variety has the topological rank 1. Consequently Lemma 4.2 implies that the topological rank of every n -allotted variety is not more than n . In particular, $r_t(\text{RA}_2^{(s)}) \leq s$.

5. Low bound for the topological rank of $\text{RA}_2^{(s)}$

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a *superalgebra* (\mathbb{Z}_2 -graded algebra) with the *even part* \mathcal{A}_0 and the *odd part* \mathcal{A}_1 , i. e. $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j \pmod{2}}$ for $i, j \in \{0, 1\}$; G be the *Grassmann algebra* on a countable set of anticommuting generators $\{e_1, e_2, \dots \mid e_i e_j = -e_j e_i\}$ with the natural \mathbb{Z}_2 -grading (G_0 and G_1 are spanned by the words of even and, respectively, odd length on $\{e_i\}$). The *Grassmann envelope* $G(\mathcal{A})$ of \mathcal{A} is the subalgebra $G_0 \otimes \mathcal{A}_0 + G_1 \otimes \mathcal{A}_1$ of the tensor product $G \otimes \mathcal{A}$. It is well known that $G(\mathcal{A})$ satisfies a multilinear identity $f = 0$ iff \mathcal{A} satisfies the certain graded identity $\tilde{f} = 0$ called the *superization* of $f = 0$. Here \tilde{f} denotes the so-called *superpolynomial corresponding to f* and we also say that \mathcal{A} satisfies the *superidentity* $\tilde{f} = 0$. The descriptions of the process of constructing of superpolynomials (the *superizing process*) can be found in [22]–[25]. For a given variety \mathcal{V} of algebras, \mathcal{A} is said to be a \mathcal{V} -*superalgebra* if $G(\mathcal{A}) \in \mathcal{V}$, i. e. if \mathcal{A} satisfies all the superizations of the defining identities of \mathcal{V} .

Let ε be one of the elements $0, 1 \in F$ and $\mathcal{A}^{(\varepsilon)} = \mathcal{A}_0^{(\varepsilon)} \oplus \mathcal{A}_1^{(\varepsilon)}$ be a superalgebra with the countable basis $x, a_{i,j}$ ($i, j = 0, 1, \dots$) such that $x \in \mathcal{A}_1^{(\varepsilon)}$ and $a_{i,j} \in \mathcal{A}_0^{(\varepsilon)}$ iff $i \equiv j \pmod{2}$. We introduce the multiplication of the basis elements of $\mathcal{A}^{(\varepsilon)}$ as follows:

$$\begin{aligned} x^2 &= \frac{\varepsilon}{2} a_{0,0}, \quad a_{i,j} \cdot x = a_{i,j+1}, \quad x \cdot a_{i,2j} = (-1)^i (a_{i,2j+1} - a_{i+1,2j}), \\ x \cdot a_{i,2j+1} &= \frac{(-1)^i}{2} (a_{i,2j+2} - 2a_{i+1,2j+1} + a_{i+2,2j}), \end{aligned}$$

and all the other products are zero. By definition, it is not hard to see that $\mathcal{A}^{(\varepsilon)}$ is a metabelian algebra.

Lemma 5.1. *The algebra $\mathcal{A}^{(\varepsilon)}$ is an RA_2 -superalgebra.*

Proof. We check that $\mathcal{A}^{(\varepsilon)}$ satisfies the superization of (1.1):

$$(a, b, c) + (-1)^{|b||c|} (a, c, b) = 0,$$

where a, b, c are homogeneous basis elements of $\mathcal{A}^{(\varepsilon)}$ and $|a|$ denotes the parity of a , i. e. $|a| = k$ for $a \in \mathcal{A}_k^{(\varepsilon)}$ ($k = 0, 1$). In view of metability of $\mathcal{A}^{(\varepsilon)}$ and the odd parity of x , it suffices to verify the relation

$$a_{i,j}([L_x, R_x] - (-1)^{i+j} L_x^2) = 0. \quad (5.1)$$

First for even j , we calculate

$$\begin{aligned} a_{i,j} [L_x, R_x] &= (-1)^i (a_{i,j+1} - a_{i+1,j}) R_x - a_{i,j+1} L_x = \\ &= (-1)^i (a_{i,j+2} - a_{i+1,j+1}) - \frac{(-1)^i}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) = \\ &= \frac{(-1)^i}{2} (a_{i,j+2} - a_{i+2,j}); \\ a_{i,j} L_x^2 &= (-1)^i (a_{i,j+1} - a_{i+1,j}) L_x = \\ &= \frac{1}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) + a_{i+1,j+1} - a_{i+2,j} = \\ &= \frac{1}{2} (a_{i,j+2} - a_{i+2,j}). \end{aligned}$$

To conclude the proof it remains to make the similar calculations for odd j :

$$\begin{aligned} a_{i,j} [L_x, R_x] &= \frac{(-1)^i}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) R_x - a_{i,j+1} L_x = \\ &= \frac{(-1)^i}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) + (-1)^{i+1} (a_{i,j+2} - a_{i+1,j+1}) = \\ &= \frac{(-1)^{i+1}}{2} (a_{i,j+2} - a_{i+2,j}); \\ a_{i,j} L_x^2 &= \frac{(-1)^i}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) L_x = \\ &= \frac{1}{2} (a_{i,j+2} - a_{i+1,j+1} + a_{i+1,j+1} - 2a_{i+2,j} + a_{i+3,j-1} + a_{i+2,j} - a_{i+3,j-1}) = \\ &= \frac{1}{2} (a_{i,j+2} - a_{i+2,j}). \quad \square \end{aligned}$$

Proposition 5.1. *The algebra $\mathcal{A}^{(\varepsilon)}$ satisfies the relations*

$$[a_{i,2j}, x]_s = a_{i+1,2j}, \quad (5.2)$$

$$[[a_{i,j}, x]_s, x]_s = a_{i+2,j}, \quad (5.3)$$

12 Alexey Kuz'min

where $[a, b]_s = ab - (-1)^{|a||b|}ba$ is a supercommutator of the elements a, b .

Proof. First we calculate

$$[a_{i,2j}, x]_s = a_{i,2j} R_x - (-1)^i a_{i,2j} L_x = a_{i,2j+1} - (a_{i,2j+1} - a_{i+1,2j}) = a_{i+1,2j}.$$

Thus (5.2) and, consequently, (5.3), for even j , are proved. Further, for odd j , we have

$$\begin{aligned} [a_{i,j}, x]_s &= a_{i,j} R_x + (-1)^i a_{i,j} L_x = a_{i,j+1} + \frac{1}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) = \\ &= \frac{3}{2} a_{i,j+1} - a_{i+1,j} + \frac{1}{2} a_{i+2,j-1}. \end{aligned}$$

Finally, combining the obtained relation with (5.2), we get

$$\begin{aligned} [[a_{i,j}, x]_s, x]_s &= \frac{1}{2} [(3a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}), x]_s = \\ &= \frac{1}{2} (3a_{i+1,j+1} - 3a_{i+1,j+1} + 2a_{i+2,j} - a_{i+3,j-1} + a_{i+3,j-1}) = a_{i+2,j}. \quad \square \end{aligned}$$

Let $\mathcal{I}^{(k)}$ ($k \in \mathbb{N}$) be the span of all elements $a_{i,j} \in \mathcal{A}^{(\varepsilon)}$ such that $i \geq 2k$. By the definition of multiplication in $\mathcal{A}^{(\varepsilon)}$, every nonzero product $a_{i,j} T_x$ is a linear combination of elements $a_{i',j'}$ such that $i' \geq i$. Consequently, $\mathcal{I}^{(k)}$ is an ideal of $\mathcal{A}^{(\varepsilon)}$. For every natural $n \geq 2$, we introduce the quotient superalgebra $\mathcal{A}^{(n)}$ as follows:

$$\mathcal{A}^{(2k)} = \mathcal{A}^{(0)} /_{\mathcal{I}^{(k)}}, \quad \mathcal{A}^{(2k+1)} = \mathcal{A}^{(1)} /_{\mathcal{I}^{(k)}}.$$

Lemma 5.2. *The algebra $\mathcal{A}^{(n)}$ is an $\text{RA}_2^{(n)}$ -superalgebra.*

Proof. Taking into account Lemma 5.1, it suffices to prove that $\mathcal{A}^{(n)}$ satisfies the superization of (1.3). By virtue of metability of $\mathcal{A}^{(n)}$, we need to verify the relation

$$\underbrace{[\dots [[\mathcal{A}^{(n)}, x]_s, x]_s, \dots, x]_s}_n = 0.$$

By the definition of $\mathcal{A}^{(n)}$, using (5.3), for $k = \lfloor \frac{n}{2} \rfloor$, we have

$$\underbrace{[\dots [[a_{i,j}, x]_s, x]_s, \dots, x]_s}_{2k} = a_{i+2k,j} = 0.$$

Thus the required relation is proved for even n . In the case of odd n , it remains to check the following:

$$\underbrace{[\dots [[x, x]_s, x]_s, \dots, x]_s}_{2k+1} = \underbrace{[\dots [[a_{0,0}, x]_s, x]_s, \dots, x]_s}_{2k} = a_{i+2k,j} = 0. \quad \square$$

Lemma 5.3. *The variety $\text{Var } \mathcal{G}(\mathcal{A}^{(n)})$ is not $(n-1)$ -allotted.*

Proof. In the case $n = 2k + 1$, we need to verify that

$$\underbrace{[\dots [\mathcal{A}^{(n)}, x]_s, x]_s, \dots, x]_s}_{2k} R_x^j \neq 0, \quad j \in \mathbb{N}.$$

Applying (5.2), we calculate

$$\underbrace{[\dots [x, x]_s, x]_s, \dots, x]_s}_{2k} R_x^j = \underbrace{[\dots [a_{0,0}, x]_s, x]_s, \dots, x]_s}_{2k-1} R_x^j = a_{2k-1,j} \neq 0.$$

For $n = 2k$, we need to check that

$$\underbrace{[\dots [(\mathcal{A}^{(n)})^2, x]_s, x]_s, \dots, x]_s}_{2k-2} R_x^j \neq 0, \quad j \in \mathbb{N}.$$

Using (5.3), we get

$$\underbrace{[\dots [a_{0,1}, x]_s, x]_s, \dots, x]_s}_{2k-2} R_x^j = a_{2k-2,j+1} \neq 0. \quad \square$$

In view of (3.5), Lemma 5.2 implies that the variety $\text{Var } G(\mathcal{A}^{(n)})$ is n -allotted. Consequently by Lemma 5.3, we have $r_t(\text{RA}_2^{(n)}) \geq n$ for $n = 2, \dots, s$. Finally, comparing this estimate with the result of Sec. 4, we obtain $r_t(\text{RA}_2^{(s)}) = s$.

Acknowledgments

This article was carried out at the Department of Mathematics and Statistics (IME) of the University of São Paulo as a part of the author's post-doc project supported by the São Paulo Research Foundation (FAPESP), grant 2010/51880-2. The author is very thankful to the IME for the kind hospitality and the creative atmosphere, to his supervisor Prof. I. P. Shestakov for his attention to this article, and to Prof. S. V. Pchelintsev for suggesting the problem and for the useful discussions on the obtained results.

References

- [1] A. R. Kemer, Finite basis property of identities of associative algebras, *Algebra Logic* **26** (1987) 362–397; translation from *Algebra Logika* **26** (1987) 597–641.
- [2] A. Kemer, Ideals of identities of associative algebras In: *AMS Translations of Mathematical Monograph*, **87**, Amer. Math. Soc., Providence, RI, 1988.
- [3] W. Specht, Gesetze in Ringen, *Math. Zeits.* **52** (1950) 557–589.
- [4] A. V. Grishin, Examples of infinitely based T-spaces and T-ideals in characteristic 2, *Fundam. Prikl. Mat.* **5** (1999) 101–118 (Russian).
- [5] A. Ya. Belov, Counterexamples to the Specht problem, *Sb. Math.* **191** (2000) 329–340; translation from *Mat. Sb.* **191** (2000) 13–24.
- [6] V. V. Schigolev, Examples of T-spaces with an infinite basis, *Sb. Math.* **191** (2000) 459–476; translation from *Mat. Sb.* **191** (2000) 143–160.

- [7] A. Ya. Vais, E. I. Zel'manov, Kemer's Theorem for finitely generated Jordan algebras, *Sov. Math.* **33** (1989) 38–47; translation from *Izv. Vyssh. Uchebn. Zaved. Mat.* **6** (1989) 42–51.
- [8] A. V. Il'tyakov, Finiteness of basis of identities of a finitely generated alternative PI-algebra over a field of characteristic zero, *Sib. Math. J.* **32** (1991) 948–961; translation from *Sib. Mat. Zh.* **32** (1991) 61–76.
- [9] A. V. Il'tyakov, On finite basis of identities of Lie algebra representations, *Nova J. Algebra Geom.* **1** (1992) 207–259.
- [10] U. U. Umirbaev, Specht varieties of soluble alternative algebras, *Algebra Logic* **24** (1985) 140–149; translation from *Algebra Logika* **24** (1985) 226–239.
- [11] V. P. Belkin, Varieties of right alternative algebras, *Algebra Logic* **15** (1976) 309–320; translation from *Algebra Logika* **15** (1976) 491–508.
- [12] I. M. Isaev, Finite-dimensional right alternative algebras that do not generate finitely based varieties, *Algebra Logic* **25** (1986) 86–96; translation from *Algebra Logika* **25** (1986) 136–153.
- [13] Yu. A. Medvedev, Finite basis theorem for varieties with a two-term identity, *Algebra Logic* **17** (1978) 458–472; translation from *Algebra Logika* **17** (1978) 705–726.
- [14] Yu. A. Medvedev, Example of a variety of solvable alternative algebras over a field of characteristic 2 having no finite basis of identities, *Algebra Logic* **19** (1980) 191–201; translation from *Algebra Logika* **19** (1980) 300–313.
- [15] A. M. Kuz'min, On Spechtian varieties of right alternative algebras, *J. Math. Sci., New York* **149** (2008) 1098–1106; translation from *Fundam. Prikl. Mat.* **12** (2006) 89–100.
- [16] S. V. Pchelintsev, On identities of right alternative metabelian Grassmann algebras, *J. Math. Sci., New York* **154** (2008) 230–248; translation from *Fundam. Prikl. Mat.* **13** (2007) 157–183.
- [17] S. V. Pchelintsev, Varieties of algebras that are solvable of index 2, *Math. USSR, Sb.* **43** (1982) 159–180; translation from *Mat. Sb.* **115(157)** (1981) 179–203.
- [18] V. S. Drensky, T. G. Rashkova, Varieties of metabelian Jordan algebras, *Serdica* **15** (1989) 293–301.
- [19] S. V. Platonova, Varieties of two-step solvable algebras of type (γ, δ) , *J. Math. Sci., New York* **139** (2006) 6762–6779; translation from *Fundam. Prikl. Mat.* **10** (2004) 157–180.
- [20] A. V. Badeev, The variety N_3N_2 of commutative alternative nil-algebras of index 3 over a field of characteristic 3, *Fundam. Prikl. Mat.* **8** (2002) 335–356 (Russian).
- [21] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Translated from the Russian by Harry F. Smith. (Academic Press, Inc., New York–London, 1982).
- [22] E. I. Zel'manov, I. P. Shestakov, Prime alternative superalgebras and nilpotency of the radical of a free alternative algebra, *Math. USSR, Izv.* **37** (1991) 19–36; translation from *Izv. Akad. Nauk SSSR, Ser. Mat.* **54** (1990) 676–693.
- [23] I. P. Shestakov, Superalgebras and counterexamples, *Sib. Math. J.* **32** (1991) 1052–1060; translation from *Sib. Mat. Zh.* **32** (1991) 187–196.
- [24] I. P. Shestakov, Prime Mal'tsev superalgebras, *Math. USSR, Sb.* **74**: (1993) 101–110; translation from *Mat. Sb.* **182** (1991) 1357–1366.
- [25] M. Vaughan-Lee, Superalgebras and dimensions of algebras, *Int. J. Algebra Comput.* **8** (1998) 97–125.